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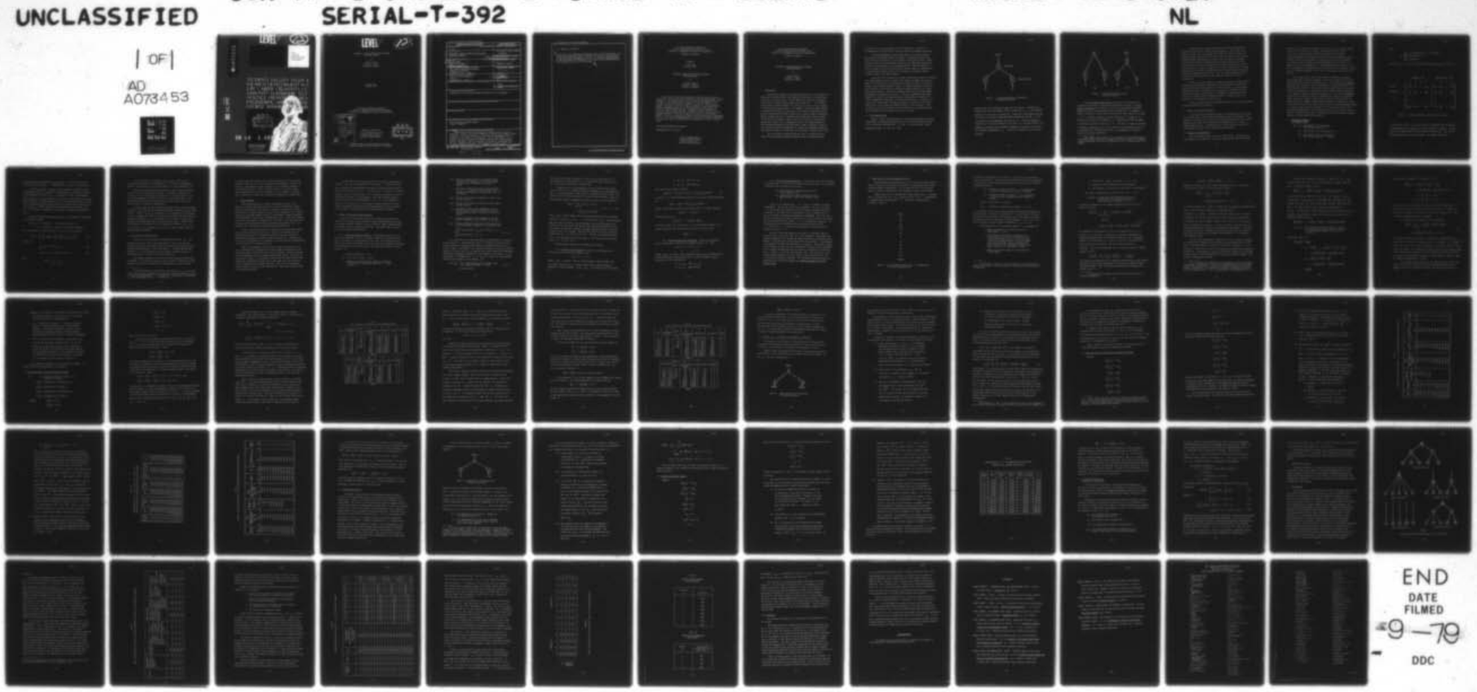

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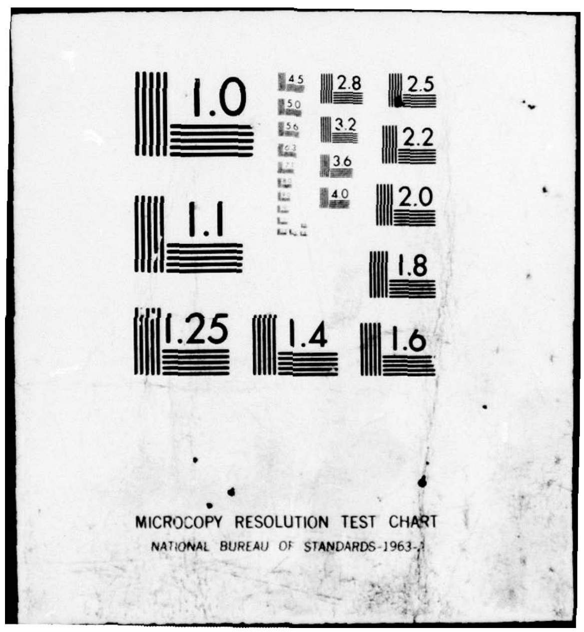
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DESIGNING A MULTI-PRODUCT MULTI-ECHELON INVENTORY SYSTEM

by

Donald Gross
Charles E. Pinkus
Richard M. Soland

Serial T-392
22 June 1979

The George Washington University
School of Engineering and Applied Science
Institute for Management Science and Engineering

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1. INTRODUCTION

Distribution systems for manufactured goods are often composed of a hierarchy of warehouses that stock goods for distribution to other warehouses and to retail stores, at which demand for these goods originates. A multi-facility distribution system such as this is generally called a multi-echelon inventory system. Of course, such systems are not limited to warehouses and retail stores. For example, the factory which produces the goods could be part of the system. The factory's inventories would include raw materials and work in process at various stages of completion. Another example would be a repairable item system where there are depot and field repair stations. Other factors that can make a distribution system more complex are transshipments, that is, the redistribution of stock between warehouses on the same level, repair facilities at some or all distribution points (combination of repair and consumable products), and exogenous demand at any facility in the system, to mention only a few.

Once the number of installations and their locations have been fixed, the multi-echelon inventory problem is one of finding the best

inventory policy for each product at each installation. There is a lot known about this problem but there are also many unanswered questions.

The problem addressed in this paper pushes the decision process back one step by assuming neither the number of installations nor their locations have been fixed for a multi-product distribution system. Organizations setting up new distribution systems or re-examining existing ones have paid little attention to this design problem. They have generally fixed the number of levels of distribution and the number of installations in some arbitrary manner and have then attempted to determine optimal stockage levels for the given system.

In Section 2 of this paper we present a mathematical model for solving this design problem; that is, a model for finding the number of echelons, number of installations, and where products should be stocked for a multi-product, multi-echelon inventory system. The model presented in Section 2 relies heavily on the ability to determine inventory policies for a given multi-echelon system. A method for finding these policies, using dynamic programming, is presented in Section 3. Section 4 generalizes the model presented in Section 2 to include capacity constraints and presents a 0-1 linear programming formulation of this generalized model. In Section 5 we present some computational experience obtained in solving the two design models given in this paper and indicate areas of future research. This introductory section concludes with an illustration of the design problem and a discussion of the premises on which it is based.

1.1 The Design Problem

Figure 1 shows a three-installation arborescent-configuration system. Installation B (warehouse) obtains goods from a source (factory) and feeds the two lower-level installations, A_1 and A_2 (retail stores), where exogenous demands, D_1 and D_2, occur.

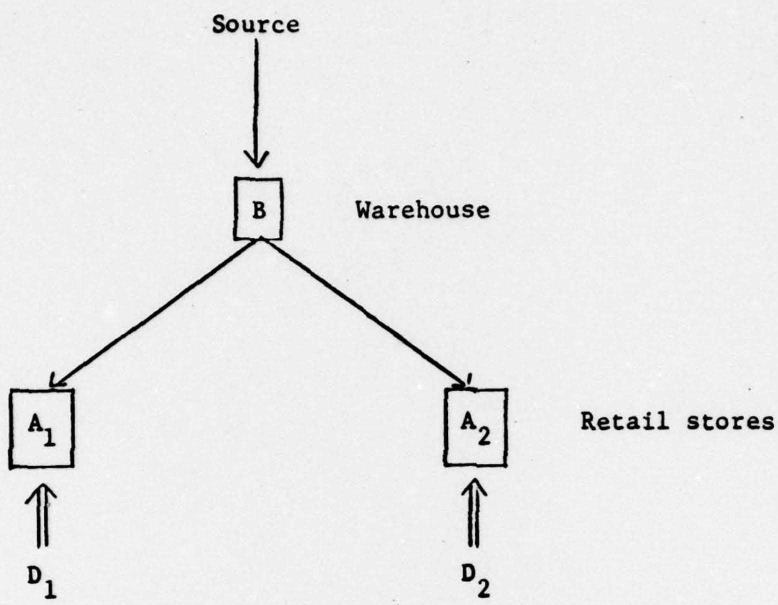

Figure 1. A three-installation, arborescent-configuration system.

Figure 2 gives two alternative designs for this system. In Figure 2(a) installation B has been removed from the system and A_1 , A_2 are supplied directly from the source of production. Figure 2(b) shows another alternative design to the basic three-installation arborescent system. A_2 has been removed and its demand goes directly to B . Since the exogenous demand is still located, physically, near where A_2 had been located, the dashed line indicates that individual shipments are made one at a time to the customers represented by D_2 . This can be thought of as mail-order business.

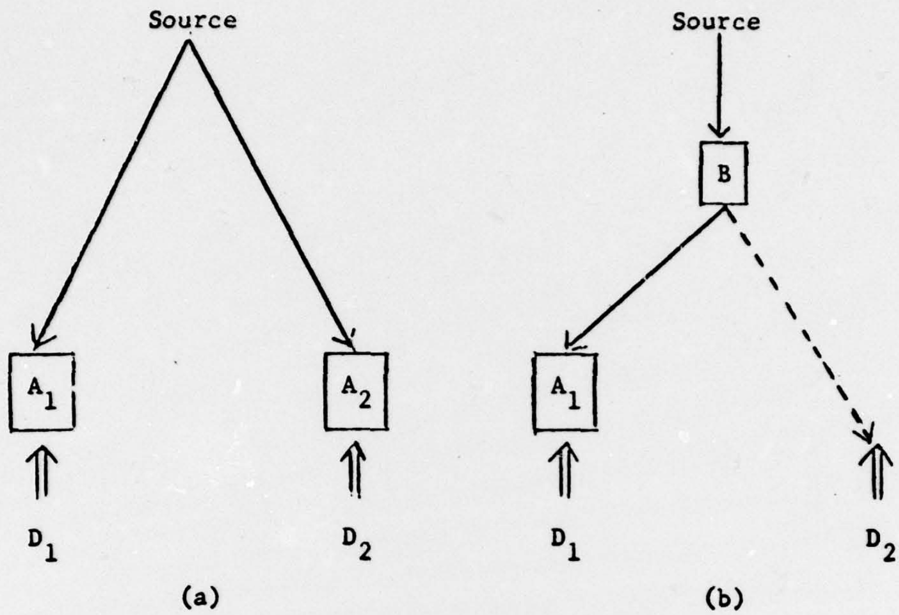

Figure 2. Alternative designs for the system of Figure 1.

The phrase echelon structures will be used in referring to the alternative designs for a multi-echelon distribution system. The basic echelon structure will be thought of as the one containing the maximum number of levels and installations under consideration. For example, if Figure 1 represents a basic echelon structure then the system has a maximum of $2^3 = 8$ possible alternative echelon structures.[1] In general, if a basic echelon structure is composed of p installations, then the number of alternative echelon structures is 2^p. Of course, it might be possible to eliminate immediately some of the structures from consideration due to the availability of existing facilities or to geographical, political, economic, and other constraints.

[1]This number would increase if we considered more situations than an installation simply being in the system or not in the system. For example, a third possibility is to have an installation carry a lifetime supply of a product.

The multi-product, multi-echelon inventory system design problem discussed in this paper is based upon two premises. First, there is no reason to suspect that the minimum cost design will be one in which all products use the same echelon structure. That is, given a set of echelon structures under consideration, if we were to find the best inventory policy for each product stocked under each echelon structure, it would very likely be found that different echelon structures were best for different products. Second, the objective to minimize some measure of total system cost will usually be accomplished only if some products do not use the echelon structure best for them. This is because different products stored at the same installation will share the fixed cost of that installation.

To illustrate these premises we have constructed an example problem concerned with designing a four-product inventory system in which four echelon structures are considered [see Pinkus (1975)]. In this example, which uses realistic inventory costs, we find that a different echelon structure is best for each product. Furthermore, to minimize the total cost of the system, two of the products cannot be stored in ways that minimize their individual inventory costs.

The next section presents a mathematical model for solving the design problem described in this section.

2. ANALYSIS OF THE DESIGN PROBLEM

In this section a mathematical model for determining the best design for a multi-product, multi-echelon distribution system is presented. The purpose of this model is to find the best echelon structure for each product, bearing in mind that the products are not independent when it comes to sharing installations. A superposition of the best echelon structure for each product will then result in the best system design.

2.1 Measure of Effectiveness

We first consider what is meant by "best" design. The measure of effectiveness to be used will be one that minimizes some measure of total

system cost, for example, total expected cost per year or total discounted cost for the lifetime of the system. The system cost will include all the real (out of pocket) operating expenses of the system, capital expenditures for building the system, and those intangible factors to which a cost can be assigned. The penalty cost for loss of goodwill as a result of back ordering (not meeting demand when it occurs) is an example of such an intangible cost. Any intangible factor to which a cost can be assigned can be handled directly by the model to be presented. Other ways of handling intangible factors will be discussed at the end of this section.

For a given product using a given echelon structure, we separate these system costs into two categories. The first includes all the costs associated with the inventory stockage policy. We call these the inventory costs and they include the costs of procurement, carrying inventory, filling orders, and stockouts. The second category includes all the costs associated with operating the installations of an echelon structure regardless of the number of products using the installations. We call these facility costs and they include the capital expenditure for building the installations, along with a number of fixed costs associated with operating the installations, for example, administrative expenses, the expense of renting facilities (if they are not built), and certain other fixed operating expenses for a given product that do not depend on the inventory policies used at the various installations. The facility costs are all fixed costs that are normally considered sunk costs in solving a multi-echelon inventory problem, because the echelon structure has been set.

2.2 Mathematical Model of the Design Problem

Let a_{ij} = the inventory cost of product j using echelon structure i ,

b_k = the facility cost of installation k ,

b_{ik} = the facility cost of installation k when echelon structure i is used,

where

$$b_{ik} = \begin{cases} b_k \text{ , if installation } k \text{ is included in} \\ \text{echelon structure } i \text{ ,} \\ 0 \text{ , otherwise.} \end{cases}$$

The costs a_{ij} and b_{ik} are arrayed in the matrix $[a_{ij} \vdots b_{ik}]$ of Figure 3.

$$\begin{array}{ll} & \begin{array}{cccccccc} \multicolumn{4}{c}{\text{Products } (j)} & \multicolumn{4}{c}{\text{Installations } (k)} \\ 1 & 2 & \cdots & n & 1 & 2 & \cdots & p \end{array} \\ \begin{array}{lc} \text{Echelon} & 1 \\ \text{Structures} & 2 \\ \quad (i) & \vdots \\ & m \end{array} & \left[\begin{array}{cccc:cccc} a_{11} & a_{12} & \cdots & a_{1n} & b_{11} & b_{12} & \cdots & b_{1p} \\ a_{21} & a_{22} & \cdots & a_{2n} & b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & & & & \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_{m1} & b_{m2} & \cdots & b_{mp} \end{array} \right] \end{array}$$

Figure 3. Array of inventory costs and facility costs.

If the facility costs b_{ik} were all zero, all products could be treated independently and the solution to this problem would be obvious. We would select the minimum element in each column of matrix $[a_{ij}]$, that is, the minimum inventory cost for each product. The echelon structure i

associated with this minimum cost for product j would be the echelon structure used by product j. Unfortunately, the b_{ik} are not all zero.

The design problem is similar to the assignment problem of linear programming in that it is desired to assign products to echelon structures so that minimum cost over the a_{ij} is achieved. Unlike the assignment problem, however, multi-assignments in any row are permitted and the assignment of products to echelon structures can create additional costs (facility cost), depending upon which installations are required in the echelon structures chosen. A further complication occurs because each facility cost, b_k, is a fixed cost, regardless of the number of products using installation k.

We now define decision variables and give a mathematical formulation for the design problem.

Let $x_{ij} = 1$ if product j uses echelon structure i,

$= 0$ if product j does not use echelon structure i.

Then the problem is to find the matrix $X = [x_{ij}]$ which minimizes

$$F(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}x_{ij} + \sum_{k=1}^{p} u\left[\sum_{i=1}^{m} (b_{ik} \sum_{j=1}^{n} x_{ij})\right] b_k \tag{1}$$

subject to:

$$\sum_{i=1}^{m} x_{ij} = 1, \; j=1,2,\ldots,n, \tag{2}$$

$$x_{ij} = 0 \text{ or } 1, \text{ for all } i \text{ and } j, \tag{3}$$

where

$$u(y) = 0 \text{ if } y \leq 0,$$

$$= 1 \text{ if } y > 0.$$

Thus the problem of designing a multi-product, multi-echelon distribution system has been formulated as a 0-1 nonlinear programming problem. The objective is to find the matrix X^* which minimizes $F(\cdot)$.

Real-life design problems often must take into account intangible factors to which costs cannot be applied. For example, suppose it is necessary to have a particular installation in the system because of certain trade agreements, regardless of the cost this might entail. This situation may be handled by considering only those echelon structures which include the particular installation.

We have solved the above mathematical model using a branch-and-bound algorithm. This algorithm is described elsewhere [see Pinkus, Gross, Soland (1973)]. However, one of the weaknesses of the formulation just presented is that it assumes that there is no limit to the storage space available at a given installation. It is one purpose of this paper to overcome this weakness by incorporating space constraints in this model. Before doing that, however, we describe one way of finding the inventory costs, a_{ij} . To obtain these costs requires the solution of another problem--the multi-echelon inventory problem.

3. MULTI-ECHELON INVENTORY MODEL

This section is concerned with obtaining the value of a_{ij} , the inventory cost to be used in the design model.[2] This value, and the associated inventory stockage policies, are arrived at by solving a multi-echelon inventory problem. Thus, for product j stocked under echelon structure i , it is desired to find the optimal inventory policies, at each installation of the structure, which yield a_{ij} .

Only dynamic solutions to the multi-echelon inventory problem were considered. This means that we want to make a number of inventory stockage decisions over time, not consider just a one-period or static solution.

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[2] To avoid confusion in this section, the notation a_{ij} will be used rather than the phrase inventory cost, where the particular echelon structure i or the particular product j is immaterial. The methods for obtaining a_{ij} will be applicable for all i and for all j .

At first glance this seems to be an obvious assumption for the inventory problem so that it will tie into the problem of designing a system to be used for a long period of time. However, it is conceivably optimal for a distribution system to stock a lifetime supply of a product, rather than reorder from the source of production. If one wanted to consider such a situation, it would best be handled by modifying the design model.

3.1 Clark's Approach

Clark (1958) presented a dynamic programming solution to the problem of finding inventory policies for a multi-installation, single-product inventory model with stochastic demand. His method for solving multi-echelon inventory problems is built on the framework of the classical approach to uncertain demand, periodic review dynamic inventory problems, first presented by Arrow, Harris, and Marschak (1951). Under this approach, the cost of the system is represented as a function of the inventory level. This cost function includes ordering, holding, and shortage costs. The objective is to minimize the total expected discounted cost.

Clark's solution was shown to be optimal by Clark and Scarf (1960) when the installations were in series. However, they pointed out that Clark's procedure was not necessarily optimal for the more realistic arborescent configuration of installations (see Figure 1). Nevertheless, they felt that under many real situations it gave an excellent approximation to the solution of the arborescent case.

One of the assumptions made by Clark and Scarf was that the cost to an installation of ordering an item from a higher installation in the system was linear, without any fixed cost of ordering. The only exception to this assumption was at the highest installation, where a fixed cost of ordering was allowed. In a subsequent paper [Clark and Scarf (1962)] they relaxed this assumption and were able to give upper and lower bounds on the cost function for the optimal solution. This led to an approximate optimal inventory policy for each installation, based upon the upper bound on the optimal solution.

A modification of the Clark approach will be used to determine the values of a_{ij} for the design model of this paper. It should be noted that any method for determining multi-echelon inventory policies could be used to find the values of a_{ij} . The selection of a method is, of course, dependent upon the characteristics of the inventory situation for which a multi-echelon system is being designed. The Clark approach has been selected because (1) the determination of the a_{ij} is an immediate by-product of the calculation of the optimal inventory policies, (2) the dynamic programming calculations are straightforward, and (3) the model is rich enough to include many multi-echelon inventory situations.

3.2 Review of Classical Inventory Models

Before describing Clark's method for finding optimal inventory policies for a single product stored in a multi-echelon system, we review the classical dynamic, periodic review, stochastic demand inventory model for a single item at a single installation. This model forms the basis for the Clark multi-echelon model.

3.2.1 Classical Inventory Problem - A purchasing decision is to be made at the beginning of each of a number of regularly spaced periods of time, for example, at the beginning of each week. This decision will be based on the level of inventory at the time, ordering, holding, and shortage costs during the period, and the effect the decision will have on future periods. Let

z = purchase quantity, $z \geq 0$

$c(z)$ = cost of purchasing z units

λ = number of time periods lag between an order and its delivery, the possible values for λ being 0,1,2,... .

$\phi(t)$ = probability density function of demand during a period, where demand is a continuous random variable and is independent from period to period

u = inventory on hand at the end of a period, where $-\infty < u < \infty$ (a negative value indicates that demand occurred during the period that could not be filled)

$h(u)$ = holding cost charged on inventory on hand at the end of a period

$p(u)$ = shortage cost charged for failure to meet demand during a period

x_n = inventory on hand at the beginning of the nth period, before an order is received, that is, the inventory on hand at the end of the previous period

y_n = inventory on hand at the beginning of the nth period, immediately after an order is received

$L(y_n)$ = expected holding and shortage cost during the nth period (hereafter referred to as the period cost).

$$= \begin{cases} \int_0^{y_n} h(y_n - t)\phi(t)dt + \int_{y_n}^{\infty} p(t-y_n)\phi(t)dt \ , & y_n \geq 0 \\ \int_0^{\infty} p(t-y_n)\phi(t)dt \ , & y_n < 0 \ . \end{cases} \qquad n = 1,2,\ldots$$

If a delivery is to be received during a given period as a result of an order placed λ periods before, then it is assumed this order arrives at the beginning of the period and before the purchase decision for the period is made. Furthermore, it is assumed the supplier carries an infinite supply of the item, that is, the supplier never back orders the installation. The dynamic programming formulation of this problem is now given. In this formulation it is assumed that $\lambda = 0$, that is, delivery is instantaneous, and that excess demand is back ordered. Let

$C_n(x_n,y_n)$ = total expected discounted inventory cost for a problem lasting n periods $\qquad n = 1,2,\ldots$

α = discount factor.

The periods are numbered backwards in time, thus, period number one is the last period of the problem. It is assumed that units on hand at the end of the last period have no salvage value.

Suppose we are at the beginning of the nth period of the problem, that is, there are n periods of business remaining for the installation, and x_n is the inventory on hand before an ordering decision is made. The optimal policy for the nth period is the policy which minimizes $C_n(x_n,y_n)$. The well-known dynamic programming recursive relation for this problem is:

$$\begin{aligned} \hat{C}_n(x_n) = \min_{y_n \geq x_n} \{ & c(y_n - x_n) + L(y_n) \\ & + \alpha \int_0^\infty \hat{C}_{n-1}(y_n - t)\phi(t)dt\} \end{aligned} \qquad n = 1,2,\ldots \qquad (4)$$

where $\hat{C}_n(x_n)$ equals minimum total expected discounted cost for a problem lasting n periods, $n = 1,2,\ldots$. In this equation $\hat{C}_n(x_n)$ has been broken down into three components: the purchasing cost for the nth period; the period cost for the nth period; and the total expected discounted cost for $n-1$ periods of operation, assuming an optimal inventory policy is followed during the last $n-1$ periods. This recursive relation is used to find the optimal value of y_n , which we call S_n . Clearly, the desired inventory level at the beginning of the nth period, S_n , has an effect on all future levels S_i , $i = 1,2,\ldots,n-1$.

Two well-known results for this model are now given.

3.2.2 No Fixed Cost of Ordering - Assume the purchasing cost is linear with no fixed cost of ordering. Then

$$c(z) = c \cdot z \ , \ z \geq 0 \ .$$

Assume $L(y_n)$ is convex. Then it has been shown by Karlin (1958) that the optimal policy for an n period problem can be characterized by a sequence of critical numbers $S_1,S_2,\ldots,S_n$. The policy for the kth period is:

$$\text{if } x_k \leq S_k \text{ , order } S_k - x_k \text{ ,}$$

$$\text{if } x_k > S_k \text{ , order nothing.}$$

This result can be shown as follows:

$$C_k(x_k,y_k) = c(y_k-x_k) + L(y_k) + \alpha \int_0^\infty \hat{C}_{k-1}(y_k-t)\phi(t)dt \ . \tag{5}$$

$\hat{C}_{k-1}(y_k-t)$ is a convex function because it is the sum of convex functions.

Let

$$W_k(y_k) = L(y_k) + \alpha \int_0^\infty \hat{C}_{k-1}(y_k-t)\phi(t)dt \ .$$

$W_k(y_k)$ is the sum of convex functions and, thus, is a convex function.

$$c(y_k-x_k) = c \cdot (y_k-x_k) \ .$$

Substituting into (5)

$$C_k(x_k,y_k) = c \cdot (y_k-x_k) + W_k(y_k) \ .$$

We desire to find S_k , the value of y_k which minimizes function C_k . Taking the partial derivative and setting equal to zero, we get

$$W_k'(S_k) = -c \ .$$

3.2.3 <u>Positive Fixed Cost of Ordering</u> - Assume the purchasing cost is linear with a fixed cost of ordering equal to K . Then

$$\begin{aligned} c(z) &= c \cdot z + K \ , \ z > 0 \ , \\ &= 0 \qquad\qquad , \ z = 0 \ . \end{aligned}$$

Assume $L(y_k)$ is convex. Scarf (1960) has shown that the optimal policy for the kth period is defined by a pair of critical numbers, (S_k,s_k) . The policy for the kth period is:

$$\text{if } x_k \leq s_k \text{ , order } S_k - x_k \text{ ,}$$

$$\text{if } x_k > s_k \text{ , order nothing.}$$

3.2.4 Two Unnecessary Assumptions - In presenting the dynamic programming formulation for the classical single-installation model, two unnecessary assumptions have been made for the sake of

(a) simplifying the following description of Clark's approach, and

(b) simplifying the computer program which has been written to obtain the inventory costs, a_{ij} .

First, it was assumed that $\lambda = 0$, that is, delivery of an order is immediate. This is not a necessary assumption. For the case where $\lambda > 0$, Karlin and Scarf (1958) show that the optimal purchase quantity is a function of the total stock on hand plus on order, regardless of the dates of delivery, assuming excess demand is backlogged. They show how to reduce a problem with an order time lag ($\lambda>0$) to one in which no lag exists, and thus the above results, where $\lambda = 0$, are applicable in general. In developing their multi-echelon results, Clark and Scarf allow for $\lambda \geq 0$ ($\lambda=0,1,2,\ldots$) . To simplify the discussion of these results we continue to assume $\lambda = 0$.

Second, it has been implied by the lack of a subscript that the functions $\phi(t)$, $c(z)$, $h(u)$, and $p(u)$ are stationary. The dynamic programming formulation for solving the single-installation problem can easily handle a situation where these functions are non-stationary. Nevertheless, the assumption that they are stationary is continued. This greatly simplifies the amount of input data and bookkeeping necessary for the operation of the computer program that has been written to determine the inventory costs. Finally, it is noted that although $\phi(t)$ was stated to be a continuous density function, all the results of this section hold if $\phi(t)$ is a discrete probability function. The following discussion assumes the continuous case, while the examples and computer program utilize the discrete case.

3.3 Multi-echelon, Series-configuration Model

Having reviewed two important cases of the classical single-echelon model, we are now in a position to describe the Clark approach to a multi-echelon stochastic demand model. For the present, only a multi-echelon system in which the installations are arranged in series is considered.

Suppose there are N installations, where installation N supplies stock to installation $N-1$, $N-1$ supplies stock to $N-2, \ldots,$ installation 2 supplies stock to installation 1. This series configuration is pictured in Figure 4.

Source

↓

[N]

↓

[N-1]

↓

⋮

↓

[2]

↓

[1]

↑↑

Demand

Figure 4. A multi-echelon system with N installations in a series configuration.

The highest installation in the series, N , receives its stock from the source of production.[3] The notation and assumptions given at the beginning of this section still generally apply and are not repeated here, with the following exceptions:

Let x_n^i = inventory on hand at echelon i at the beginning of the nth period before an order is received

y_n^i = inventory on hand at echelon i at the beginning of the nth period, immediately after an order is received

$L_i(y_n^i)$ = period cost of echelon i

$c_i(z)$ = ordering cost function for echelon i , $z \geq 0$.

It is important to note the following distinction between an installation and an echelon. The stock at installation i refers only to the stock physically at that location. But when we refer to the stock at echelon i , we mean the sum of all the stocks at installations i, i-1,...,2,1 plus all the stocks in transit between installations i, i-1,...,2,1 .

The formulation of the Clark model follows. It is assumed that:

1. Demand, exogeneous to the system, occurs at installation 1 only.

2. The purchasing cost between installations is linear without a fixed cost of ordering. This cost can be thought of as the cost to transport a unit from one installation to the next installation in the system. The only exception to this assumption is at the highest installation, where a fixed cost of ordering is allowed.

3. Demand in excess of supply at any installation is backlogged.

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[3] The source of production could be included in the series as the highest installation. Then, its source of stock would be the supplier of raw material.

4. The functions $L_i(y_n^i)$ are convex, $i = 1,2,\ldots,N$.
5. Delivery at each installation is instantaneous.[4]
6. The functions $c(z)$, $h(u)$, $p(u)$ and $\phi(t)$ are stationary.[4]

The model is described for the special case $N = 2$. Let

$\hat{C}_n(x_n^1,x_n^2)$ = minimum total expected discounted cost of the system at the beginning of period n before an order is placed, $n = 1,2,\ldots$.

Following the approach for the single-installation model, we get the recursive relation:

$$\hat{C}_n(x_n^1,x_n^2) = \min_{\substack{x_n^1 \le y_n^1 \le y_n^2 \\ y_n^2 \ge x_n^2}} \{c_1(y_n^1-x_n^1) + c_2(y_n^2-x_n^2) \qquad n = 1,2,\ldots$$

$$+ L_1(y_n^1) + L_2(y_n^2) + \alpha \int_0^\infty \hat{C}_{n-1}(y_n^1-t,\ y_n^2-t)\phi(t)dt\}$$

The problem with this approach is that in the general case $\hat{C}_n$ is a function of N variables. Therefore, the recursive calculations of dynamic programming would be prohibitively long, even for a high-speed computer, if the function $\hat{C}_n$ is left in this form. Clark and Scarf have overcome this difficulty. They prove that the function $\hat{C}_n(x_n^1,x_n^2,\ldots,x_n^N)$ can be decomposed into N functions, each of a single variable, that is,

$$\hat{C}_n(x_n^1,x_n^2,\ldots x_n^N) = \hat{D}_n^1(x_n^1) + \hat{D}_n^2(x_n^2) + \ldots + \hat{D}_n^N(x_n^N) .$$

This permits the computation of the optimal inventory level at each echelon separately using the same method that is used for the classical single-installation model. What is involved is described by returning to the special case, $N = 2$.

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[4]It was pointed out earlier in this section that this is not a necessary assumption.

$$\hat{C}_n(x_n^1,x_n^2) = \hat{D}_n^1(x_n^1) + \hat{D}_n^2(x_n^2) \ , \ n = 1,2,\ldots \ .$$

Begin by considering the lower echelon and assume there is no limitation on the stock available to it from the higher echelon.

$$\hat{D}_n^1(x_n^1) = \min_{y_n^1 \geq x_n^1} \{c_1(y_n^1-x_n^1) + L_1(y_n^1)$$

$$+ \alpha \int_0^\infty \hat{D}_{n-1}^1(y_n^1-t)\phi(t)dt\} \ , \ n = 1,2,\ldots \ . \qquad (6)$$

It has been assumed that $L_1(y_n^1)$ is convex and the purchasing (transportation) cost is linear without a fixed cost of ordering, that is, $c_1(y_n^1-x_n^1) = c_1 \cdot (y_n^1-x_n^1)$. Therefore, recursive relation (6) is identical to Equation (4), and the optimal policy for this echelon is described by the sequence of critical numbers $S_1^1,S_2^1,\ldots,S_n^1$, where S_i^1 is the optimal level of inventory for echelon 1 in the ith period. This conclusion has assumed installation 2 can satisfy the demand of installation 1. This will not always be the case. However, Clark and Scarf show that installation 2 should satisfy as much of the demand from installation 1 as is possible and backlog the rest.[5]

The fact that installation 2 might not be able to satisfy the demand of installation 1 suggests that in solving for the optimal policy of echelon 2, a shortage penalty, in addition to the penalty included in L_2 , be incurred by installation 2 when it must backlog the demand that installation 1 places on it. This penalty is simply the additional expected cost suffered at installation 1 because installation 2 could not satisfy its demand. This penalty is determined as follows.

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[5] It is assumed that installation 2 will deliver at most one shipment each period to installation 1 and that this shipment goes out in the "split second" after installation 2 has made its reorder decision but before its stock is replenished. This sequence of decisions is necessary in order to force this example, with $\lambda = 0$, to behave like a real problem with $\lambda > 0$.

Consider the kth period at echelon 1. Suppose $x_k^1 < S_k^1$. Then echelon 1 will order $S_k^1 - x_k^1$ stock from echelon 2, and the total cost for k periods at echelon 1 will be

$$\hat{D}_k^1(x_k^1) = c_1 \cdot (S_k^1 - x_k^1) + L_1(S_k^1) + \alpha \int_0^\infty \hat{D}_{k-1}^1(S_k^1 - t)\phi(t)dt .$$

If the stock level at echelon 2 is such that $x_k^2 < (S_k^1 - x_k^1)$, that is, installation 2 cannot satisfy all the demand from installation 1, then echelon 1 will have x_k^2 stock on hand after ordering, rather than S_k^1 stock. This means that the order received by installation 1 is $x_k^2 - x_k^1$. In other words, all the available stock at installation 2 was shipped to installation 1. Under this situation, the total cost for k periods at echelon 1 will be

$$\bar{D}_k^1(x_k^1, x_k^2) = c_1 \cdot (x_k^2 - x_k^1) + L_1(x_k^2) + \alpha \int_0^\infty \hat{D}_{k-1}^1(x_k^2 - t)\phi(t)dt .$$

Let $\Delta_n^i(\cdot)$ = the additional shortage penalty at echelon i for not being able to meet demand at echelon i-1 during the nth period, $i = 2,3,\ldots,N$, $n = 1,2,\ldots$.

The penalty $\Delta_k^2(\cdot)$ is

$$\begin{aligned}
&\bar{D}_k^1(x_k^1, x_k^2) - \hat{D}_k^1(x_k^1) \\
&\quad = c_1 \cdot (x_k^2 - x_k^1) - c_1 \cdot (S_k^1 - x_k^1) + L_1(x_k^2) - L_1(S_k^1) \\
&\qquad + \alpha \int_0^\infty [\hat{D}_{k-1}^1(x_k^2 - t) - \hat{D}_{k-1}^1(S_k^1 - t)]\phi(t)dt \\
&\quad = c_1 \cdot (x_k^2 - S_k^1) + L(x_k^2) - L(S_k^1) \\
&\qquad + \alpha \int_0^\infty [\hat{D}_{k-1}^1(x_k^2 - t) - \hat{D}_{k-1}^1(S_k^1 - t)]\phi(t)dt \\
&\quad = \Delta_k^2(x_k^2) .
\end{aligned}$$

A more complete statement of the function Δ_n^2 is

$$\Delta_n^2(x_n^2) = c_1 \cdot (x_n^2 - S_n^1) + L_1(x_n^2) - L_1(S_n^1)$$
$$+ \alpha \int_0^\infty [\hat{D}_{n-1}^1(x_n^2 - t) - \hat{D}_{n-1}^1(S_n^1 - t)]\phi(t)dt \ ,$$
$$\text{for } x_n^2 \le S_n^1 \ , \ n = 1,2,\ldots$$
$$= 0 \ , \text{ for } x_n^2 > S_n^1 \ , \ n = 1,2,\ldots \ .$$

The fact that Δ_n^2 is a function of x_n^2 alone is very significant. It means that the recursive relation for the total cost of n periods of operation at echelon 2 is a function of the single variable x_n^2. Thus, a dynamic programming solution is feasible.

The optimal policy at echelon 2 is now described.

$$\hat{D}_n^2(x_n^2) = \min_{y_n^2 \ge x_n^2} \{c_2(y_n^2 - x_n^2) + L_2(y_n^2) + \Delta_n^2(x_n^2)$$
$$+ \alpha \int_0^\infty \hat{D}_{n-1}^2(y_n^2 - t)\phi(t)dt\} \ , \ n = 1,2,\ldots \ . \tag{7}$$

It has been assumed that $L_2(y_n^2)$ is convex and the purchasing cost is linear with a fixed cost of ordering, that is, $c_2(y_n^2 - x_n^2) = c_2 \cdot (y_n^2 - x_n^2) + K$. Since Δ_n^2 is the sum of convex functions, it is a convex function. Therefore, the solution of (7) follows the solution given for the classical model with a fixed cost of ordering, and the optimal policy for the nth period at echelon 2 is described by the pair of critical numbers (S_n^2, s_n^2), where the superscript indicates the echelon.

The procedure described here for two echelons can be used for any number of echelons in series. Clark and Scarf show that the policies obtained in this way are optimal policies, and their proof allows for $\lambda > 0$.

A summary of the important ideas behind the Clark approach follows.

1. The decision variable is the stock level of an echelon, not the stock level of an installation.

2. An N installation problem in N decision variables becomes N separate problems, each in one decision variable. Each separate problem is a special case of the classical single-installation stochastic demand inventory problem and can be solved by dynamic programming.

3. The cost function for each echelon i (except the first) includes the additional shortage penalty Δ_n^i for not being able to supply the complete order of echelon i-1 in the nth period. This covers the additional expected cost resulting from echelon i-1 not being able to achieve its optimal level of stock for the period. The penalty Δ_n^i is obtained from the solution of echelon i-1.

We now present an example of a two-installation problem. The optimal policies are calculated for two periods.

3.4 Two-installation, Series-configuration Example

Let u_i = inventory on hand in echelon i at the end of a period.

$h_i^*(u_i)$ = holding cost at installation i.

$h_i(u_i)$ = holding cost at echelon i.

$p_i^*(u_i)$ = shortage cost at installation i.

$p_i(u_i)$ = shortage cost at echelon i.

Suppose

$$h_1^*(u_1) = 2.2u_1$$

$$h_2^*(u_2) = 2.0u_2$$

$$p_1^*(u_1) = 72u_1$$

$$p_2^*(u_2) = 5u_2$$

$$c_1(z) = 5z$$

$$c_2(x) = 50z + 30$$

$$\alpha = 1 \ ,$$

where all costs are in dollars.

Since this model determines optimal echelon policies, the period cost L_i , $i = 1,2$, must be based upon echelon costs, not installation costs. The holding and shortage costs at echelon 2 are merely these respective costs at installation 2. Thus,

$$h_2(u_2) = h_2^*(u_2) = 2u_2 \ , \text{ and}$$

$$p_2(u_2) = p_2^*(u_2) = 5u_2 \ .$$

To find the holding cost at echelon 1, it is noted that any units on hand at echelon 1 at the end of a period have already been charged \$2 holding cost because they were counted in the stock of echelon 2 that was on hand at the end of the same period. The same reasoning applies to the penalty cost at echelon 1. Thus,

$$h_1(u_1) = h_1^*(u_1) - h_2^*(u_1) = 2.2u_1 - 2.0u_1 = 0.2u_1 \ , \text{ and}$$

$$p_1(u_1) = p_1^*(u_1) - p_2^*(u_1) = 72u_1 - 5u_1 = 67u_1 \ .$$

If there were N installations arranged in series, the holding and shortage costs at echelon N would be obtained the same way as for echelon 2 in this example, and the costs for echelons $N-1, N-2, \ldots, 1$ would be obtained using the cost-added concept that was used for echelon 1 in this example. Note that the cost-added concept implies that $h_{n-1}^* \geq h_n^*$ and $p_{n-1}^* \geq p_n^*$ for $n = 2,3,\ldots N$.

Suppose the demand $\phi(t)$ for this example follows a Poisson distribution, and the mean demand is one unit per period. The period cost is found by evaluating the following expression:

$$L_i(y_n^i) = \sum_{t=0}^{y_n^i} h_i \cdot (y_n^i - t)\phi(t) + \sum_{t=y_n^i+1}^{\infty} p_i \cdot (t - y_n^i)\phi(t) \ , \ y_n^i \geq 0 \ ,$$

$$i = 1,2, \quad n = 1,2,\ldots$$

$$= \sum_{t=0}^{\infty} p_i \cdot (t - y_n^i)\phi(t) \ , \ y_n^i < 0 \ , \quad i = 1,2, \quad n = 1,2,\ldots \ .$$

Note that the functions $L_i(y_n^i)$, $i = 1,2$, satisfy the assumptions of convexity because the holding and shortage cost functions are convex.

We now evaluate the recursive relations $\hat{D}_n^1$ and $\hat{D}_n^2$ for $n = 1$ and $n = 2$, from which the optimal policies will be obtained. The results appear in Tables 1 and 2, which are read from right to left. This follows the way dynamic programming is used to find the optimal policy of the last period (period 1) and then moves backwards in time, calculating successively the optimal policies for periods $2,3,\ldots,n$. Wherever possible the notation defined earlier is used to head the columns of these tables. Each table is split into two parts, the lower, representing echelon 1, and the upper, representing echelon 2.

Table 1 shows the various possible costs for period 1 at different stock levels. The actual interpretation of stock level [Columns (1) and (8)], that is, whether it is stock on hand before or after receiving an order, depends on the cost column under consideration. Columns (2) and (9) of Table 1 show zero cost because this is the last period of operation and there are no future costs. Successive differences of the entries in Column (4) are shown in Column (5). These are used to locate the critical number S_1^1 . It was shown in the description of the classical single-installation model without a fixed cost of ordering that the critical

Table 1

PERIOD 1 SOLUTION OF TWO-INSTALLATION, SERIES-CONFIGURATION EXAMPLE

$\hat{D}_1^2(x_1^2)$ (15)	$\hat{D}_1^2$ Assuming Always Order When $x_1^2 < S_1^2$ (14)	$W_1^2(x_1^2) - W_1^2(x_1^2-1)$ (13)	$W_1^2(\cdot)$ Total Expected Cost Before Ordering (12)	$L_2(y_1^2)$ (11)	$\Delta_1^2(x_1^2)$ (10)	$\alpha \sum_{t=0}^{\infty} \hat{D}_0^2(y_1^2-t)\phi(t)$ (9)	Stock Level (8)
12.00	12.00	2.00	12.00	12.00	0.00	0.00	7
10.00	10.00	2.00	10.00	10.00	0.00	0.00	6
8.00	8.00	1.97	8.00	8.00	0.00	0.00	5
6.03	6.03	1.87	6.03	6.03	0.00	0.00	4
4.16	4.16	1.25	4.16	4.16	0.00	0.00	3
2.91	2.91	-12.41	2.91	2.72	0.19	0.00	2
15.32	15.32	-39.70	15.32	2.57	12.75	0.00	1
55.02	55.02	-67.00	55.02	5.00	50.02	0.00	0
122.02	135.02		122.02	10.00	112.02	0.00	-1
185.02	185.02		189.02	15.00	174.02	0.00	-2
235.02	235.02		256.02	20.00	236.02	0.00	-3
285.02	285.02		323.02	25.00	298.02	0.00	-4

ECHELON 2

$\Delta_1^2(x_1^2)$ (7)	$\hat{D}_1^1(x_1^1)$ (6)	$W_1^1(x_1^1) - W_1^1(x_1^1-1)$ (5)	$W_1^1(\cdot)$ *Total Expected* Cost Before Ordering (4)	$L_1(y_1^1)$ (3)	$\alpha \sum_{t=0}^{\infty} \hat{D}_0^1(y_1^1-t)\phi(t)$ (2)	*Stock* Level (1)
0.00	1.20	.20	1.20	1.20	0.00	7
0.00	1.00	.16	1.00	1.00	0.00	6
0.00	0.84	-.04	0.84	0.84	0.00	5
0.00	0.88	-1.08	0.88	0.88	0.00	4
0.00	1.96	-5.19	1.96	1.96	0.00	3
0.19	6.96		7.15	7.15	0.00	2
12.75	11.96		24.70	24.70	0.00	1
50.02	16.96		66.98	66.98	0.00	0
112.02	21.96		133.98	133.98	0.00	-1
174.02	26.96		200.98	200.98	0.00	-2
236.02	31.96		267.98	267.98	0.00	-3
298.02	36.96		334.98	334.98	0.00	-4

ECHELON 1

number S_n satisfied $W(S_n) = -c$. Since we have shifted to discrete valued functions, the method of finite differences must be used to find S_n^1 . It can be shown that the critical number S_n^1 satisfies the relation

$$W_n^1(S_n^1) - W_n^1(S_n^1-1) \leq -c_1 \leq W_n(S_n^1+1) - W_n^1(S_n^1) . \tag{8}$$

Successive differences are shown in Column (5) until the inequality (8) is satisfied. In the case of echelon 1 during period 1 ,

$$-5.19 \leq -c_1 = -5 \leq -1.08 ,$$

thus $S_1^1 = 3$.

Column (6) of Table 1 shows the total expected cost after ordering, for various stock levels before ordering. Since it is optimal to have a stock level of three units at the beginning of period one, if $x_1^1 < 3$, we purchase $3 - x_1^1$ units and add the cost of this purchase, $5 \cdot (3-x_1^1)$, to $\hat{D}_1^1(3)$. Column (7) shows the cost of not being able to bring the stock level up to S_1^1 . This cost, Δ_1^2 , is equal to Column (4) minus Column (6) and is the penalty charged to echelon 2 as a result of installation 2 back-ordering installation 1.

The echelon 2 section of Table 1 shows the costs involved in obtaining the critical numbers (S_1^2, s_1^2). Much of this section is similar to the echelon 1 section. Only the differences are commented upon. Column (10) is equal to Column (7). In Column (12) the function W_1^2 includes the additional cost Δ_1^2 . Column (13) shows that $S_1^2 = 0$. Column (14) is obtained by assuming an order is always placed to bring the stock level up to zero if $x_1^2 < 0$. In comparing Columns (14) and (12), note that it is not always wise to order when $x_1^2 < 0$. When $x_1^2 = -1$, the total cost after ordering is \$135.02, while if we had not ordered, the total cost would

have been \$122.02. The stock level, s_1^2 , at which the costs in Column (14) are less than or equal to the costs in Column (12), is the level at which ordering should begin. In this case, $s_1^2 = -2$. Column (15) shows the total expected cost after ordering, for various stock levels before ordering. The entries of Column (15) are the minimum of the entries in Columns (12) and (14).

Table 2 shows the determination of the critical numbers for period 2. Columns (6) and (15) of Table 1 have been brought back one period in time to become Columns (2) and (9), respectively, of Table 2. The calculations in Table 2 are obtained as described for Table 1.

The critical numbers for the last two periods of this problem are:

$$S_1^1 = 3 \ , \ (S_1^2, s_1^2) = (0,-2)$$

$$S_2^1 = 3 \ , \ (S_2^2, s_2^2) = (2,0) \ .$$

If we are interested in using this two-installation, series-configuration example in the design of a multi-echelon system, which would be in business for only two periods, and if there are zero units on hand at each installation at the beginning of period 2 before an order is placed, then a_{ij} for this echelon structure is

$$\hat{D}_2^1(0) + \hat{D}_2^2(0) = \$23.92 + \$165.95 = \$189.87 \ .$$

The calculation of the critical numbers for this example were carried out for 20 periods. The critical numbers for the 20th period are

$$S_{20}^1 = 5 \ , \ (S_{20}^2, s_{20}^2) = (7,1) \ .$$

The total expected discounted (inventory) cost of this echelon structure for this product, that is, a_{ij} , when it is assumed zero units are on hand at each installation before an order is placed at the beginning of the 20th period, is

Table 2

PERIOD 2 SOLUTION OF TWO-INSTALLATION, SERIES-CONFIGURATION EXAMPLE

$\hat{D}_2^2(x_2^2)$ (15)	$\hat{D}_2^2$ Assuming Always Order When $x_2^2 < S_2^2$ (14)	$W_2^2(x_2^2) - W_2^2(x_2^2-1)$ (13)	$W_2^2(\cdot)$ Total Expected Cost Before Ordering (12)	$L_2(y_2^2)$ (11)	$\Delta_2^2(x_2^2)$ (10)	$\alpha \sum_{t=0}^{\infty} \hat{D}_1^2(y_2^2-t)\phi(t)$ (9)	Stock Level (8)
22.02	22.02	3.91	22.02	12.00	0.00	10.02	7
18.11	18.11	3.53	18.11	10.00	0.00	8.11	6
14.58	14.58	2.00	14.58	8.00	0.00	6.58	5
12.58	12.58	-2.95	12.58	6.03	0.00	6.55	4
15.53	15.53	-20.42	15.53	4.16	0.00	11.37	3
35.95	35.95	-53.70	35.95	2.72	5.19	28.04	2
89.65	115.95		89.65	2.57	22.74	64.34	1
165.95	165.95		189.22	5.00	65.02	119.20	0
215.95	215.95		322.26	10.00	132.02	180.24	-1
265.95	265.95		449.04	15.00	199.02	235.02	-2
315.95	315.95		571.04	20.00	266.02	285.02	-3
365.95	365.95		693.04	25.00	333.02	335.02	-4

ECHELON 2

$\Delta_2^2(x_2^2)$ (7)	$\hat{D}_2^1(x_2^1)$ (6)	$W_2^1(x_2^1) - W_2^1(x_2^1-1)$ (5)	$W_2^1(\cdot)$ Total Expected Cost Before Ordering (4)	$L_1(y_2^1)$ (3)	$\alpha \sum_{t=0}^{\infty} \hat{D}_1^1(y_2^1-t)\phi(t)$ (2)	Stock Level (1)
0.00	2.28	0.19	2.28	1.20	1.08	7
0.00	2.11	-0.40	2.11	1.00	1.11	6
0.00	2.51	-1.77	2.51	0.84	1.67	5
0.00	4.28	-4.64	4.28	0.88	3.40	4
0.00	8.92	-10.19	8.92	1.96	6.96	3
5.19	13.92		19.11	7.15	11.96	2
22.74	18.92		41.66	24.70	16.96	1
65.02	23.92		88.94	66.98	21.96	0
132.02	28.92		160.94	133.98	26.96	-1
199.02	33.92		232.94	200.98	31.96	-2
266.02	38.92		304.94	267.98	36.98	-3
333.02	43.92		376.94	334.98	41.96	-4

ECHELON 1

$$\hat{D}^1_{20}(0) + \hat{D}^2_{20}(0) = \$1438.17 \ .$$

The series configuration of installations does not represent a very realistic multi-echelon system. If the Clark approach is to be used to obtain the a_{ij} for the design model, it will have to be applied to the more realistic arborescent configuration of installations. The next section shows how this approach is used to find the critical numbers and the a_{ij} for installations arranged in arborescence and extends the method to allow for exogenous demand at any installation.

3.5 Finding a_{ij} for Arborescent-configuration Structures

The purpose of this section is to describe how the Clark approach presented in the previous sections is applied to a multi-echelon situation in which the installations are arranged in an arborescent structure.

Figure 5 shows the simplest multi-echelon, arborescent-configuration system, where the two installations at the lowest level of the system, A_1 and A_2, are fed by installation B .

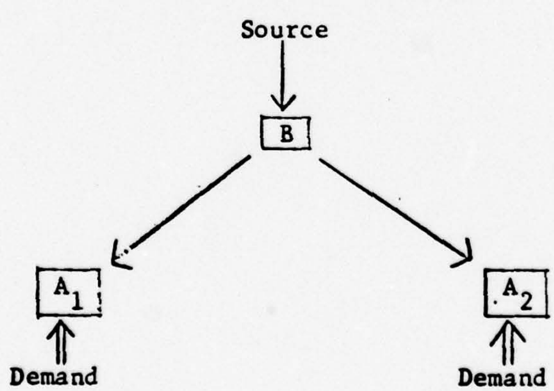

Figure 5. Three-installation, arborescent-configuration system.

The situation illustrated in Figure 5 will be used to describe how Clark's approach is applied to arborescent configurations.

The assumptions in the previous section for the series-configuration system apply here with the following exception. Demand, exogenous to the system, occurs at all the installations at the lowest level of the system, not at only one installation. The notation defined in the previous section will be used in this section. A more cumbersome method of labeling the installations is employed so that the level as well as the installation is easy to identify.

Using Clark's approach to calculate the inventory policies for the kth period for each echelon of the system, we would proceed as follows.

(1) Assume installations A_1 and A_2 are independent. Determine the inventory policy for A_1 by applying the same method that was used for echelon 1 in the series example, that is, assume stock is available at B to satisfy the order from A_1. Also, calculate Δ_k^{B,A_1}, the additional cost that would be experienced by installation A_1 if installation B cannot satisfy the demand of A_1 during the kth period.

(2) Repeat Step (1) for installation A_2, that is, calculate its inventory policy assuming B can satisfy the demand of A_2. Calculate Δ_k^{B,A_2}.

(3) Construct the additional shortage penalty Δ_k^B from the marginal per unit costs associated with Δ_k^{B,A_1} and Δ_k^{B,A_2}. The penalty Δ_k^B is an additional period cost for echelon B during the kth period, in the same manner that the penalty Δ_k^2 applied to echelon 2 of the example in the previous section.

(4) Determine the inventory policy for echelon B. Since a fixed cost of ordering is allowed at installation B, the highest installation in this system, the method for determining the two critical numbers (S_k^B, s_k^B) follows the method used for echelon 2 of the series-configuration example.

In short, the method outlined above says to calculate the inventory policies for each echelon separately by the methods used for the classical single-installation situation. This procedure can be extended easily to apply to a situation where B feeds more than two installations or to a system containing more than two levels.

Recall that $\hat{C}_n(x_n^{A_1}, x_n^{A_2}, x_n^B)$ is the minimum total expected discounted cost for the entire system of this example at the beginning of period n before an order is placed. If the inventory policies calculated by this method are to minimize the total expected discounted cost of the system, it is necessary that

$$\hat{C}_n(x_n^{A_1}, x_n^{A_2}, x_n^B) = \hat{D}_n^{A_1}(x_n^{A_1}) + \hat{D}_n^{A_2}(x_n^{A_2}) + \hat{D}_n^B(x_n^B) \ . \tag{9}$$

Unfortunately, it has been shown by Clark and Scarf (1960) that decomposition (9) does not always hold for the arborescent configuration of installations. They give an argument for such a decomposition by assuming that the stock levels at the lowest installations, in this case A_1 and A_2, are not out of balance. By this they mean that for a given period the ratios of stock on hand after ordering to expected demand are approximately the same for A_1 and A_2. In practical situations they feel that the stock levels at the lowest installations are rarely out of balance, and therefore they conclude that Clark's approach gives excellent approximations, if not optimal solutions, for the best inventory policies of an arborescent configuration.[6]

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[6] The notations $\hat{C}$ and $\hat{D}$ will continue to be used in the remainder of this section, even though it cannot be proven that these are the minimum costs.

If the holding, shortage, and transportation costs per unit at each of the lowest level installations are of the same order of magnitude, then it is felt that the stock levels at the lowest installations generally will be in balance. Experience with Clark's approach for obtaining inventory policies of installations arranged in an arborescent configuration indicates that his method is probably quite good.[7]

An assumption was made which should not be overlooked. It was assumed in Step (1) that installations A_1 and A_2 were independent. This means that transshipments between A_1 and A_2 are not allowed. Thus, an installation will be allowed to receive stock only from its designated supplier at a level higher in the system.

Clark's approach is now used to obtain the inventory policies for a three-installation, arborescent-configuration problem.

3.6 Three-installation, Arborescent-configuration Example

Suppose

$$h^*_{A_1}(u_{A_1}) = 2.2u_{A_1}$$

$$h^*_{A_2}(u_{A_2}) = 2.1u_{A_2}$$

$$h^*_B(u_B) = 2.0u_B$$

$$p^*_{A_1}(u_{A_1}) = 72u_{A_1}$$

$$p^*_{A_2}(u_{A_2}) = 69u_{A_2}$$

$$p^*_B(u_B) = 5.0u_B$$

[7] Clark (1960) used this approach to obtain the inventory policies that were used in the simulation of a large arborescent-configuration system. In personal conversations with him he indicated that the policies produced for this simulation were excellent.

$$c_{A_1}(z) = 5z$$

$$c_{A_2}(z) = 3z$$

$$c_B(z) = 50z + 30$$

$$\alpha = 1 \ .$$

Using the cost-added concept described in the series-configuration example, the echelon holding and shortage costs are:

$$h_{A_1}(u_{A_1}) = 0.2u_{A_1}$$

$$h_{A_2}(u_{A_2}) = 0.1u_{A_2}$$

$$h_B(u_B) = 2.0u_B$$

$$p_{A_1}(u_{A_1}) = 67u_{A_1}$$

$$p_{A_2}(u_{A_2}) = 64u_{A_2}$$

$$p_B(u_B) = 5u_B \ .$$

Suppose the demands at installations A_1 and A_2 each follow a Poisson distribution, with mean demand one unit per period. The demand experienced by echelon B is the convolution of the demands experienced by the two installations fed by B. Therefore, the demand at echelon B follows a Poisson distribution, with a mean demand of two units per period.

The calculations for the inventory policies at each echelon for period 1 are now described. We follow the four-step procedure given above.

(1) The costs and demand distribution for echelon A_1 are identical to those given for echelon 1 of the series-configuration example. Therefore, the echelon 1 section of Table 1, Columns (1) - (7), gives the calculations for period 1 of echelon A_1 in this example and $S_1^{A1} = 3$. Column (7) of Table 1 is $\Delta_1^{B,A1}$.

(2) Table 3 gives the calculations for period 1 of echelon A_2 . The arrow indicates the location of the critical number and $S_1^{A2} = 3$.

(3) We now show how $\Delta_1^{B,A1}$ and $\Delta_1^{B,A2}$ are used to construct Δ_1^{B} . If installation B cannot satisfy the demand from A_1 and A_2 , it has a choice of backordering A_1 , A_2 , or both A_1 and A_2 . It desires a plan for backordering that will penalize it the least. For example, if B is short one unit in the first period, it is clear by examining Column (7) in Table 1 and Column (7) in Table 3 that it is cheaper to backorder A_1 one unit, where the penalty is \$0.19, than to backorder A_2 one unit, where the penalty is \$2.04. Similarly, if B were short three units in period 1, it has the following four alternatives:

(a) backorder A_1 three units at a cost of \$50.02

(b) backorder A_1 two units and A_2 one unit at a cost of \$12.75 + \$2.04 = \$14.79

(c) backorder A_1 one unit and A_2 two units at a cost of \$0.19 + \$15.87 = \$16.06, or

Table 3

PERIOD 1 SOLUTION FOR ECHELON A_2 IN THE ARBORESCENT-CONFIGURATION EXAMPLE

Δ_1^{B,A_2} (7)	$\hat{D}_1^{A_2}(x_1^{A_2})$ (6)	$W_1^{A_2}(x_1^{A_2}) - W_1^{A_2}(x_1^{A_2}-1)$ (5)	$W_1^{A_2}(\cdot)$ Total Expected Cost before Ordering (4)	$L_{A_2}(y_1^{A_2})$ (3)	$\alpha \sum_{t=0}^{\infty} \hat{D}_0^{A_2}(y_1^{A_2}-t)\phi_{A_2}(t)$ (2)	Stock Level (1)
0.00	0.60	0.10	0.60	0.60	0.00	7
0.00	0.50	0.06	0.50	0.50	0.00	6
0.00	0.44	-0.13	0.44	0.44	0.00	5
0.0[illegible]	0.57	-1.12	0.57	0.57	0.00	4
0.00	1.69	→ -5.04	1.69	1.69	0.00	3
2.04	4.69		6.73	6.73	0.00	2
15.87	7.69		23.56	23.56	0.00	1
53.29	10.69		63.98	63.98	0.00	0
114.29	13.69		127.98	127.98	0.00	-1
175.29	16.69		191.98	191.98	0.00	-2
236.29	19.69		255.98	255.98	0.00	-3
297.29	22.69		319.98	319.98	0.00	-4

(d) backorder A_2 three units at a cost of $53.29.

Thus, B would choose alternative (b) in order to minimize the cost of not being able to bring the lower installations up to their desired stock levels. It is assumed that installation B will use the minimum cost backordering alternative, and furthermore, that B generally has a choice regarding how much it will backorder each lower-level installation. To determine the penalty Δ_1^B for all possible backordering situations, we first calculate the marginal per unit costs associated with Δ_1^{B,A_1} and Δ_1^{B,A_2}, called marginal Δ_1^{B,A_1} and marginal Δ_1^{B,A_2} in Table 4. These give the additional cost for backordering one more unit at A_1 and A_2, respectively. The marginal penalties are then ranked in value, starting with the smallest. The ranked penalties are then successively added to form Δ_1^B. This procedure guarantees that the least cost combination in backordering the lower-level installation is used. These calculations are shown in Table 4. The positive ranked marginal Δ_1^{B,A_1} and Δ_1^{B,A_2} start at stock level 5 because if echelon B has less than six units, then either A_1 or A_2 are below their desired level of three units each.

(4) Using the $\Delta_1^B(\cdot)$ from Table 4, the inventory policy for echelon B at the beginning of period 1 can be determined. The calculations are shown in Table 5 and follow the calculations for echelon 2 of the series-configuration example, given in Table 1. The critical numbers as indicated by the arrows are $(S_1^B, s_1^B) = (0,-2)$.

Table 4

DETERMINATION OF Δ_1^B FROM Δ_1^{B,A_1} AND Δ_1^{B,A_2} FOR THREE-INSTALLATION, ARBORESCENT-CONFIGURATION EXAMPLE

Stock Level	Δ_1^{B,A_1}	Marginal Δ_1^{B,A_1}	Δ_1^{B,A_2}	Marginal Δ_1^{B,A_2}	Ranked Marginal Δ_1^{B,A_1} and Δ_1^{B,A_2}	Δ_1^B
7	0.00	0.00	0.00	0.00	0.00	0.00
6	0.00	0.00	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	0.00	0.19	0.19
4	0.00	0.00	0.00	0.00	2.04	2.23
3	0.00	0.00	0.00	0.00	12.56	14.79
2	0.19	0.19	2.04	2.04	13.83	28.62
1	12.75	12.56	15.87	13.83	37.28	65.90
0	50.02	37.28	53.29	37.42	37.42	103.32
-1	112.02	62.00	114.29	61.00	61.00	164.32
-2	174.02	62.00	175.29	61.00	61.00	225.32
-3	236.02	62.00	236.29	61.00	61.00	286.32
-4	298.02	62.00	297.29	61.00	61.00	347.32

Table 5

PERIOD 1 SOLUTION FOR ECHELON B IN THE ARBORESCENT-CONFIGURATION EXAMPLE

$\hat{D}_1^B(x_1^B)$ (8)	$\hat{D}_1^B$, Assuming Always Order When $x_1^B < S_1^B$ (7)	$W_1^B(x_1^B) - W_1^B(x_1^B-1)$ (6)	$W_1^B(\cdot)$ Total Expected Cost before Ordering (5)	$L_E(y_1^B)$ (4)	Δ_1^B (3)	$\alpha \sum_{t=0}^{\infty} \hat{D}_0^B(y_1^B-t)\phi_B(t)$ (2)	Stock Level (1)
10.00	10.00	1.96	10.00	10.00	0.00	0.00	7
8.04	8.04	1.69	8.04	8.04	0.00	0.00	6
6.35	6.35	-.41	6.35	6.16	0.19	0.00	5
6.76	6.76	-11.56	6.76	4.53	2.23	0.00	4
18.32	18.32	-14.09	18.32	3.53	14.79	0.00	3
32.41	32.41	-39.44	32.41	3.79	28.62	0.00	2
71.85	71.85	-41.47	71.85	5.95	65.90	0.00	1
113.32	113.32	-66.00	113.32	10.00	103.32	0.00	0
179.32	193.32		179.32	15.00	164.32	0.00	-1
243.32	243.32		245.32	20.00	225.32	0.00	-2
293.32	293.32		311.32	25.00	286.32	0.00	-3
343.32	343.32		377.32	30.00	347.32	0.00	-4

If interest was only in one period of operation for this three-installation arborescent-configuration example, and if the on-hand inventory at each installation at the beginning of period 1 before ordering is zero units, then the total expected discounted cost of the system would be

$$\hat{D}_1^{A_1}(0) + \hat{D}_1^{A_2}(0) + \hat{D}_1^{B}(0) = \$16.96 + \$10.69 + \$113.32 = \$140.97 .$$

This would be the inventory cost a_{ij} for the design model. The calculations given here for one period were carried out for 20 periods. The critical numbers for the 20th period, that is, with 20 periods of business remaining, are

$$S_{20}^{A_1} = 5 \ , \ S_{20}^{A_2} = 5 \ , \ (S_{20}^{B}, s_{20}^{B}) = (11,3) .$$

The total expected discounted cost for 20 periods is \$2681.29, and this would represent the inventory cost, a_{ij} , in the design model if we were designing a system to last for 20 time periods.

3.7 Shadow Installations

The method just described for obtaining near optimal inventory policies for N installations having an arborescent configuration will be satisfactory for many of the echelon structures to be considered by the design model. But there is one situation which cannot be handled directly by the Clark approach for arborescent configurations. This is the situation in which one or more of the lowest-level installations, where exogenous demand exists, is not included in the echelon structure. This would be the situation if one or both of installations A_1 or A_2 were removed from the system of Figure 5. For example, Figure 6 shows this system with installation A_2 removed. Although installation A_2 is not physically included in the echelon structure of Figure 6, it is represented on the diagram by dashed lines because demand still exists where this installation would be located if it were included in the structure. We call installation A_2 a shadow installation. Any time an echelon structure does not include one of the lowest-level installations, that installation is called a shadow installation.

In the echelon structure of Figure 6, demand D_2 must be satisfied by installation B on an individual basis, that is, it is a "mail-order" demand.[8]

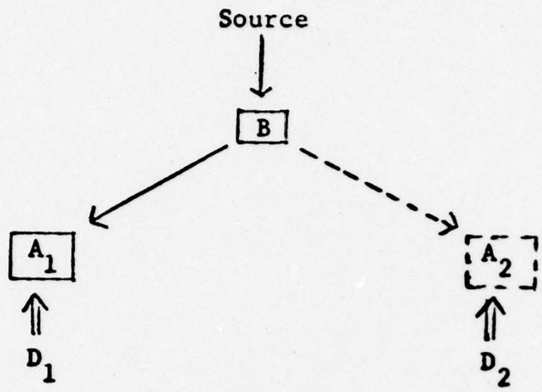


Figure 6. Arborescent configuration with a shadow installation.

Suppose we desire to determine the inventory policies for echelons A_1 and B , taking into account that mail-order demand D_2 exists at the location where installation A_2 would be placed if it were included in the system. To do this, we need a method which includes the transportation cost from installation B to the customers represented by D_2 , and a penalty cost at B for not satisfying the demand at D_2 . There is no need for a holding cost at A_2 since this installation is not in the system. Let

$\bar{p}_i^*$ = the shortage cost per unit for customers at the shadow installation i.

$\bar{c}_i$ = the transportation cost per unit in getting an item to the customers of shadow installation i from their supplier.

[8] We use the phrase "mail-order" in referring to these individual demands, but the manner in which they are processed is completely arbitrary, as long as they are processed on an individual basis. Therefore, if the product is of great enough importance to the customer, he will probably telephone the demand to installation B and request an air express delivery.

The echelon structure of Figure 6 is used to describe the method for obtaining the inventory policies and a_{ij} for a multi-echelon configuration with a shadow installation. The procedure for the kth period is:

(1) Assume demand D_2 can only be satisfied by installation B, that is, it cannot be satisfied by installation A_1. Determine the inventory policy for A_1 and Δ_k^{B,A_1} using Clark's approach for arborescent configurations.

(2) No calculations are required for echelon A_2, since installation A_2 does not exist.

(3) To construct Δ_k^B, it is necessary to consider a penalty at echelon B for not being able to satisfy the demand D_2, as well as the demand from A_1. Thus, we need something similar to the marginal per unit cost of Δ_k^{B,A_2}. Whenever a customer from demand D_2 is backordered by installation B, he incurs a cost of $p_{A_2}^*$. But part of this cost has already been applied by p_B^*; therefore, using the cost-added concept, we let $\bar{p}_{A_2}$ equal the cost per unit added by penalty $p_{A_2}^*$. Then, marginal $\Delta_k^{B,A_2} = \bar{p}_{A_2}$.

(4) The inventory policy for echelon B is determined in the same way that it was found in the arborescent configuration, with the following exception. The period cost, $L_B(\cdot)$, must include the additional cost of meeting the expected demand of D_2, λ_2, or in this case

$$L_B(y_k^B) = \bar{c}_{A_2}\lambda_2 + \sum_{t=0}^{y_k^B} h_B(y_k^B - t)\phi_B(t)$$

$$+ \sum_{t=y_k^B+1}^{\infty} p_B(t-y_k^B)\phi_B(t) \ , \ y_k^B \geq 0 \ , \ k = 1,2,\ldots$$

$$= \bar{c}_{A_2}\lambda_2 + \sum_{t=0}^{\infty} p_b(t-y_k^B)\phi_B(t) \ , \ y_k^B < 0 \ , \ k = 1,2,\ldots \ .$$

This procedure is now used to obtain the inventory policies for a numerical example of the arborescent configuration with a shadow installation shown in Figure 6.

3.8 Shadow Installation Example

Suppose

$$h_{A_1}^*(u_{A_1}) - 2.2u_{A_1}$$

$$h_B^*(u_B) = 2.0u_B$$

$$p_{A_1}^*(u_{A_1}) = 72u_{A_1}$$

$$\bar{p}_{A_2}^* = 78$$

$$p_B^*(u_B) = 5u_B$$

$$c_{A_1}(z) = 5z$$

$$\bar{c}_{A_2} = 10$$

$$c_B(z) = 50z + 30$$

$$\alpha = 1 \ .$$

Using the cost-added concept, the echelon holding and shortage costs are:

$$h_{A_1}(u_{A_1}) = 0.2u_{A_1}$$

$$h_B(u_B) = 2.0u_B$$

$$p_{A_1}(u_{A_1}) = 67u_{A_1}$$

$$\bar{p}_{A_2} = 73$$

$$p_B(u_B) = 5u_B \; .$$

Suppose the demands D_1 and D_2 are Poisson, with mean demand one unit per period.

The calculations of the inventory policies at echelons A_1 and B for period 1 are now described, following the procedure given for multi-echelon configurations with a shadow installation.

(1) The costs and demand distribution for echelon A_1 are identical to those given for echelon 1 of the series-configuration example. Therefore, the echelon 1 section of Table 1, Columns (1) - (7), gives the calculations for period 1 of echelon A_1 in this example and $S_1^{A1} = 3$. Column (7) of Table 1 is $\Delta_1^{B,A1}$.

(2) Installation A_2 is not included in the configuration, therefore S_1^{A2} is not calculated.

(3) Δ_1^B is constructed for this example using the same method applied in the three-installation arborescent-configuration example, with the exception that the marginal $\Delta_1^{B,A2} = \bar{p}_{A_2} = 73$, for all stock levels. The

ranking of the marginal $\Delta_1^{B,i}$, $i = A_1$ and A_2 , and the construction of Δ_1^B are shown in Table 6. The positive marginal Δ_1^{B,A_2} begin at stock level 0 because we expect a mail-order demand from D_2 of one unit per period and, therefore, want to penalize echelon B for holding less than one unit of stock. The positive ranked marginal Δ_1^{B,A_1} and Δ_1^{B,A_2} start at stock level 3 because if echelon B has less than four units, then either A_1 will be below its desired stock level of three units or the expected demand of one unit at D_2 will not be met.

(4) Using the $\Delta_1^B(\cdot)$ from Table 6, the inventory policy for echelon B at the beginning of period 1 can be determined. The one exception to the method used for the arborescent-configuration example is that $L_B(\cdot)$ must include the additional cost of meeting the expected demand of D_2 for each value of stock level. This cost equals the product of the expected demand and the transportation cost, $\bar{c}_{A_2}$, or in this case \$10. The costs and distribution for demand at echelon B in this example are identical to those used for echelon B of the three-installation arborescent-configuration example. Therefore $L_B(\cdot)$ for this example is the $L_B(\cdot)$ found in Table 5 plus \$10. The calculations for echelon B follow the calculations for echelon 2 of the series configuration example, given in Table 1. The critical numbers are $(S_1^B, s_1^B) = (1,-1)$.

The determination of the inventory policies for periods 2,3,... , follow the method described for period 1. The calculations for this example have been carried out 20 periods, and the critical numbers for the 20th period are:

Table 6

DETERMINATION OF Δ_1^B FOR ARBORESCENT-CONFIGURATION EXAMPLE WITH A SHADOW INSTALLATION

Stock Level	Δ_1^{B,A_1}	Marginal Δ_1^{B,A_1}	Marginal Δ_1^{B,A_2}	Ranked Marginal Δ_1^{B,A_1} and Δ_1^{B,A_2}	Δ_1^B
7	0.00	0.00	0.00	0.00	0.00
6	0.00	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.19	0.19
2	0.19	0.19	0.00	12.56	12.75
1	12.75	12.56	0.00	37.28	50.02
0	50.02	37.28	73.00	62.00	112.02
-1	112.02	62.00	73.00	62.00	174.02
-2	174.02	62.00	73.00	62.00	236.02
-3	236.02	62.00	73.00	62.00	298.02
-4	298.02	62.00	73.00	62.00	360.02

$$S_{20}^{A_1} = 5 \quad \text{and} \quad (S_{20}^{B}, s_{20}^{B}) = (9,2) \ .$$

If there are zero units on hand at each installation before ordering in the 20th period, the total expected discounted cost for 20 periods is \$2708.11. This would represent the inventory cost, a_{ij} , in the design model if we were designing a distribution system to last for 20 periods and were interested in this echelon structure. Note that the total expected discounted cost of operating the three-installation arborescent-configuration was \$2681.29. Therefore, in comparing these two echelon structures, we might conclude that the mail-order operation is a poor way to do business for this situation. But this does not take into account the inventory costs for other products or the fixed cost of an installation.

4. THE DESIGN PROBLEM WITH STORAGE SPACE CONSTRAINTS

In this section we generalize the design problem formulated in Section 2 to allow for storage space constraints at the various facilities. The resulting model yields a 0-1 linear programming problem, which includes the previously discussed design problem without capacity constraints as a special case.

Formulation of the design problem with capacity constraints requires notation in addition to that given in Section 2, and, for convenience, we give here all notation used in the model. The echelon structures are indexed by $i, i=1,\ldots,m$; the products are indexed by $j, j=1,\ldots,n$; the installations are indexed by $k, k=1,\ldots,p$. Let

a_{ij} = the inventory cost of product j using echelon structure i,

b_k = the facility cost of installation k ,

r_k = the storage space available at installation k ,

d_{ijk} = the storage space required at installation k for product j when product j uses echelon structure i.

The d_{ijk} values could be measured on the basis of either the maximum space required (over time) or the average space required (over time). The former seems more appropriate for this type of design problem, and the r_k values must then be determined accordingly. Note that the d_{ijk}, $k=1,\ldots,p$, values are obtained as outputs of the multi-echelon inventory problem solved in calculating a_{ij}. Thus $d_{ijk} = 0$ for all $j=1,\ldots,n$ if echelon structure i does not use installation k.

The decision variables are:

$$x_{ij} = 1 \text{ if product } j \text{ uses echelon structure } i\,,$$
$$= 0 \text{ otherwise,}$$
$$y_k = 1 \text{ if installation } k \text{ is used,}$$
$$= 0 \text{ otherwise.}$$

The problem, henceforth called problem (P), is then to find x_{ij} and y_k values that

$$\text{minimize} \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_{ij} + \sum_{k=1}^{p} b_k y_k \tag{10}$$

subject to

$$\sum_{i=1}^{m} x_{ij} = 1\,, \quad j=1,\ldots,n\,, \tag{11}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ijk}x_{ij} - r_k y_k \leq 0\,, \quad k=1,\ldots,p\,, \tag{12}$$

$$x_{ij}, y_k = 0 \text{ or } 1 \text{ for all } i,j,k\,. \tag{13}$$

Comparison of this formulation with that of Section 2 shows that the objective functions (1) and (10) are equivalent and constraints (2) and (11) are the same. The constraints (12) serve to impose the storage space limitations at the same time that they force the use of an installation if any echelon structure using it is chosen. If $y_k = 0$ and $d_{ijk} > 0$, then x_{ij} must be 0 in order to satisfy (12). Note that the previous formulation without storage space constraints can be easily obtained as a special case of

(10) - (13) by letting d_{ijk} equal 1 if echelon structure i uses installation k (and 0 otherwise) and setting all r_k equal to n.

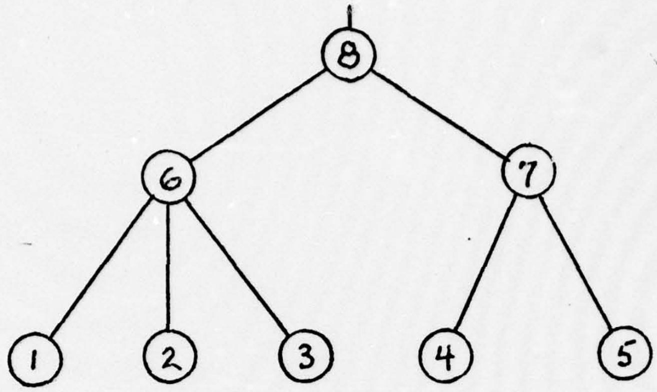
The design problem with storage space constraints has been formulated in this section as a 0-1 LP problem. In the next section we illustrate its use for designing a multi-product, multi-echelon inventory distribution system.

5. ILLUSTRATION OF MODELS

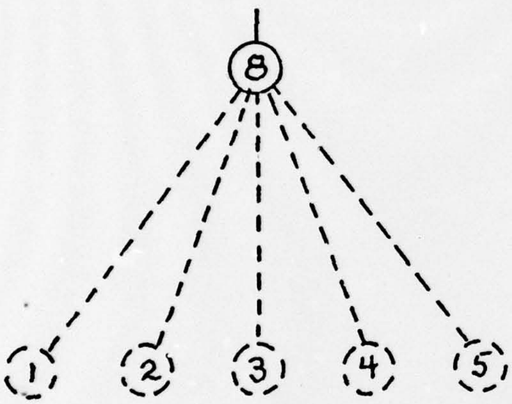
This section presents illustrative computational results using the three models given in this paper--the dynamic programming model to find optimal inventory policies for a multi-echelon distribution system, the design model without storage space constraints, and the design model with storage space constraints. The section concludes with a discussion of some problems yet to be resolved.

5.1 Illustration

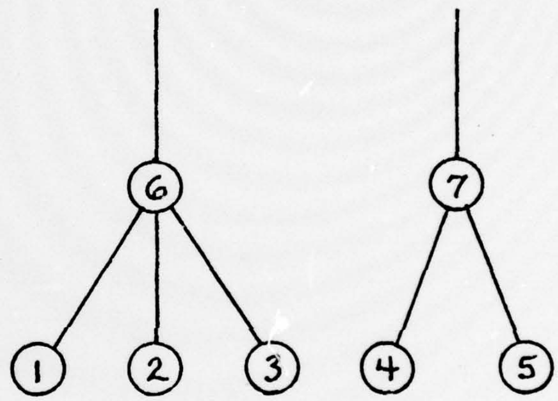
In this illustration five *alternative echelon* structures, which include up to three echelons and eight facilities, are being considered for the design of a four-product, multi-echelon inventory distribution system. The five echelon structures are shown in Figure 7. Echelon structure 1 is the full system, that is, it consists of the maximum number of facilities under consideration (a central warehouse, two regional warehouses, and five retail stores) arranged in three echelons. Although eight facilities generate $2^8 = 256$ possible echelon structures, only the four additional echelon structures shown are being considered for this system. Echelon structure 2 is a one-echelon system with mail order delivery from the central warehouse to customers located where the retail stores would be if they were in the system. Echelon structure 3 is a two-echelon system without a central warehouse. In this case the two regional warehouses are supplied directly by the sources of production. Echelon structure 4 is a single echelon system where the retail stores are supplied with products directly from the sources of production, while echelon structure 5 is similar to echelon structure 1, with the exception that customers at retail stores 1, 2, and 3 are supplied by mail order from regional



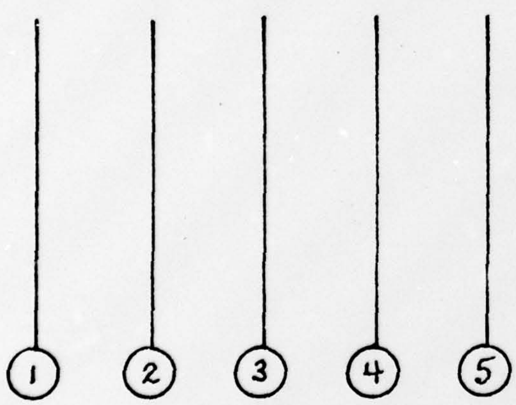
Echelon Structure 1



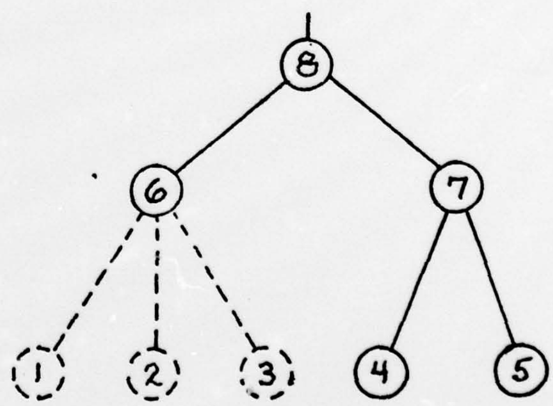
Echelon Structure 2



Echelon Structure 3



Echelon Structure 4



Echelon Structure 5

Figure 7

The Five Echelon Structures Used in the Illustration

warehouse 6.

The dynamic programming approach, described in Section 3, was used to find the multi-echelon inventory policies and the resulting a_{ij}'s for the four products of this example. Table 7 summarizes the input data needed to obtain these policies. It is assumed that the stock levels of this inventory system are reviewed quarterly and the demand for each product has a Poisson distribution, with the mean demand per quarter assumed to be the same at each retail store. If echelon structure 1 is used to stock a product, it is generally assumed that the cost to transport the product from the central warehouse (facility 8) to one of the retail stores is 6% of the procurement cost, that is, the cost to get the product to facility 8.[9] Referred to in Table 7 as "normal" transportation cost, this cost is divided evenly between the cost to move the product between facility 8 and one of the regional warehouses (facility 6 or 7) and the cost to move the product between the regional warehouse and one of the retail stores.

In the case of echelon structures with shadow installations, the transportation cost to satisfy mail-order demands will generally increase significantly. Table 7 shows these costs as a percentage of the normal transportation costs described above. Also shown are the changes in transportation cost when either the central warehouse or regional warehouse are not used to stock a given product. For products 1 and 2 we assume under such a two-echelon system that there would be no change from the normal transportation cost, whereas with products 3 and 4 it is assumed that the transportation costs double (200% of normal cost) under these circumstances. Table 7 also gives the holding and stockout penalty costs used in this illustration. The holding costs per quarter range from 2% to 8% of the total cost to get the product to the facility where this holding cost is incurred. The shortage costs at the retail stores

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[9] In two cases, for products 2 and 4 transported to retail stores 4 and 5, this cost is assumed to be 3% of the procurement cost.

Table 7

INPUT DATA NEEDED TO FIND MULTI-ECHELON INVENTORY POLICIES FOR FOUR-PRODUCT EXAMPLE.

Product	Mean Demand at each Retail Store (Units per quarter)	Costs per Unit						
		Procurement Cost ($)	Normal Transportation Cost to Retail Stores 1-3 (% of Procurement Cost)	Normal Transportation Cost to Retail Stores 4 and 5 (% of Procurement Cost)	Mail Order Transportation Cost (% of Normal Transportation Cost)	Transportation Cost without Central or Regional Warehouses (% of Normal Transportation Cost)	Holding Cost (% of Procurement plus Transportation Costs)	Stockout Penalty Cost at Retail Stores (% of Procurement plus Transportation Costs)
1	0.2	7000	6	6	1500	100	5	150
2	0.2	9000	6	3	1500	100	2	110
3	0.2	5000	6	6	500	200	8	190
4	1.3	700	6	3	1500	200	8	190

(or shadow installations) vary from 110% to 190% of the cost to get a product to these stores. At facilities other than the retail stores, a shortage cost equal to 10% of the facility's holding cost is applied to cover the nuisance created by backordering.

Finally, the following data or characteristics were assumed to be common to all the products:

1. The life of the system is 20 quarters (five years), that is, each product is stocked for 20 quarters,

2. there is no inventory on hand at the beginning of the first period of operation,

3. the interest rate is 1-1/2% per quarter, resulting in a discount factor of 0.985, and

4. the fixed cost of ordering is $100.

Table 8 gives the dynamic programming results. The a_{ij} represent the total expected discounted cost for stocking product j under echelon structure i for 20 quarters, and the inventory policies shown are for the 20th quarter. The computer program written to generate these results also gives the order policies at each facility for each of the 20 periods. It is interesting to note that steady state policies were generally reached quickly. With the exception of three of the 20 results, the policy shown for the 20th quarter was reached by the fifth quarter, and sometimes sooner.

Two types of inventory policies, S and (S,s), are shown in Table 8. At the highest level facility in a system (the facility that orders the product from the source of production) it is assumed that a fixed cost of ordering is incurred, hence (S,s) type policies result. Since there is no fixed cost of ordering at other facilities, S type policies are found. The time to calculate these critical numbers and to find the inventory costs, a_{ij}, averaged 30 seconds per product for a given echelon structure on an IBM 360/50 computer.

Next we determine the best design for this four-product system, using the model given in Section 2, that is, we assume there are no constraints on the amount of space available at each of the facilities

Table 8

INVENTORY POLICIES AND RESULTING a_{ij} FOR FOUR-PRODUCT EXAMPLE

Echelon Structure (i)	Product	a_{ij}	Total Expected Discounted Inventory Cost (a_{ij}) for 20 Quarters ($ x 1000)	S or (S,s) Inventory Policy at Facility k							
				1	2	3	4	5	6	7	8
1	1	a_{11}	195	2	2	2	2	2	6	4	(5,4)
2	1	a_{21}	235	-	-	-	-	-	-	-	(1,0)
3	1	a_{31}	196	2	2	2	2	2	(3,2)	(2,1)	-
4	1	a_{41}	199	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	-	-	-
5	1	a_{51}	191	-	-	-	2	2	1	4	(3,2)
1	2	a_{12}	219	3	3	3	3	3	6	6	(5,4)
2	2	a_{22}	272	-	-	-	-	-	-	-	(1,0)
3	2	a_{32}	221	3	3	3	3	3	(3,2)	(2,1)	-
4	2	a_{42}	224	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	-	-	-
5	2	a_{52}	229	-	-	-	3	3	1	6	(3,2)
1	3	a_{13}	155	2	2	2	2	2	6	4	(5,4)
2	3	a_{23}	143	-	-	-	-	-	-	-	(1,0)
3	3	a_{33}	160	2	2	2	2	2	(3,2)	(2,1)	-
4	3	a_{43}	167	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	-	-	-
5	3	a_{53}	128	-	-	-	2	2	1	4	(3,2)
1	4	a_{14}	104	6	6	6	6	6	15	12	(20,14)
2	4	a_{24}	139	-	-	-	-	-	-	-	(6,5)
3	4	a_{34}	107	6	6	6	6	6	(12,8)	(8,5)	-
4	4	a_{44}	112	(4,2)	(4,2)	(4,2)	(4,2)	(4,2)	-	-	-
5	4	a_{54}	112	-	-	-	6	6	4	12	(10,9)

where products might be stored. The inventory costs, a_{ij} , given in Table 8, and the facility costs, b_{ik} , for this problem are arrayed by echelon structure i in Figure 8. The facility costs are consistent with values found in previous research [see Pinkus (1971)] and take into account the fact that the fixed cost of operating a warehouse, in proportion to the value of goods that might be stocked there, is less than the fixed cost of operating a retail store, due to economies of scale and lower property values.

The solution to this problem is to stock all four products under echelon structure 5, which means that each product will be stocked at retail stores 4 and 5 and at warehouses 6, 7, and 8. The total (5-year) expected discounted cost of this solution is $766,000. It is interesting to note that of the four products, only products 1 and 3 use the echelon structure that minimizes their inventory cost. If products 2 and 4 were to use echelon structure 1, which minimizes their inventory cost, the fixed cost of the system would increase because products 2 and 4 would now be stocked at retail stores 1 - 3, as well as at facilities 4 - 8. Under this solution the savings in inventory cost do not compensate for the increased facility cost, and the total cost of the system is $790,000.

The solution to this problem was obtained in 11 seconds on an IBM 370/148 using the branch-and-bound algorithm described in Pinkus, Gross, and Soland (1973). Although initially written to handle 10 echelon structures, 30 products, and 30 facilities, the computer program for this algorithm can easily be expanded to handle larger values of m , n , and p .

Finally, we have solved the same problem with storage space constraints using the model presented in Section 4. The storage space available at each facility, r_k , is given in Table 9, and the amount of storage space required for each product is given in Table 10. To find d_{ijk} , we multiply the per unit storage space required to stock product j (see Table 10) by the maximum number of units (S) that will be stocked at facility k when echelon structure i is used (see Table 8).

ECHELON STRUCTURE (i)	PRODUCT (j)				FACILITY (k)							
	1	2	3	4	1	2	3	4	5	6	7	8
1	195	219	155	104	14	14	14	14	14	28	19	31
2	235	272	143	139	0	0	0	0	0	0	0	31
3	196	221	160	107	14	14	14	14	14	28	19	0
4	199	224	167	112	14	14	14	14	14	0	0	0
5	191	229	128	112	0	0	0	14	14	28	19	31

Figure 8

Array of Inventory Costs and Facility Costs for Four-product Example.

Table 9

STORAGE SPACE AVAILABLE
AT EACH FACILITY

Facility k	r_k (sq. feet)
1	3500
2	3500
3	3500
4	2000
5	2000
6	7000
7	5000
8	8000

Table 10

STORAGE SPACE REQUIRED BY
EACH PRODUCT

Product (j)	Storage Space Required Per Unit (sq. feet)
1	400
2	300
3	500
4	300

For example, $d_{1,1,1}$ is 800 (that is, 400 x 2), $d_{3,2,6}$ is 900 (that is, 300 x 3), and $d_{5,4,8}$ is 3000 (that is, 300 x 10).

The solution to the problem with space constraints is to stock products 1, 2, and 4 under echelon structure 4, and product 3 under echelon structure 2, that is, products 1, 2, and 4 are stocked only at the retail stores, which are supplied directly by the sources of production, and product 3 is stocked only at the central warehouse (facility 8), from which customers receive product 3 by mail order. The total (5-year) expected discounted cost for this solution is $779,000. In this case, none of the products are stocked under the echelon structure that minimizes the inventory cost for a given product. This solution was obtained in 145 seconds using a 0-1 linear programming computer code run on an HP 3000.

5.2 Discussion

We conclude this paper with a discussion of some problems yet to be resolved.

The design model with storage space constraints, problem (P), has $mn + p$ 0-1 variables and $n + p$ constraints, so the problem dimensions may be fairly high for problems of practical size. For example, with $m = n = 30$ and $p = 20$, problem (P) has 920 variables and 50 constraints. The 0-1 LP computer code used to solve the design problem with storage space constraints in the previous section can handle only 40 variables and 20 constraints. This fact, together with the special structure of problem (P), suggests that a specialized algorithm for its solution would be much more efficient for practical problems than the general integer linear programming algorithm that was used to get the results in Section 5.1. Such a special algorithm is presently being developed and tested.

There is another aspect of this work that needs to be examined. It can be shown that the model with space constraints, problem (P), might not obtain the best solution to this design problem. This is because we have restricted ourselves to using the optimal inventory policy and resulting

a_{ij} for a given product stocked under a given echelon structure. With the introduction of space constraints, a situation can occur where by making a reduction in the optimal value of S for a given product, we might be able to satisfy a space constraint that would not otherwise be satisfied. Such a reduction in S would yield a sub-optimal (greater) value for a_{ij}, but this increased inventory cost could be compensated for by the reduction in facility cost resulting from enabling the product to be squeezed into a set of facilities used by other products.

There is also the possibility of letting the decision variables, x_{ij}, take on fractional values. A fractional x_{ij} means that part of product j is stocked under one echelon structure, and the remainder under one or more other echelon structures. It is easy to see that the x_{ij} would always be 0 or 1 for the design problem without storage space constraints, but it is not clear that this should be the case for problem (P). We are in the process of allowing for the use of fractional x_{ij}, for the space constrained design problem, and plan to investigate several heuristic solution methods in order to allow for the possiblity of using sub-optimal inventory policies to reduce the total cost of the system.

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